



On the Boundedness of Solution of the Second Order Ordinary Differential Equation with Involution

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Article info

Original: 7 August 2020
Revised: 14 septembre 2020
Accepted: 26 September 2020
Published online: 20
December 2020

Key Words: Ordinary differential equation with involution; Stability; Boundedness; Existence and uniqueness

Abstract

In the present paper, the initial value problem for the second order ordinary differential equation with involution is investigated. We obtain equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution is proved. Theorem on existence and uniqueness of bounded solution of initial value problem for second order ordinary nonlinear differential equation with involution is established.

1. Introduction

Differential equations with involution appear in mathematical models of ecology, biology, and population dynamics (see, e.g, [1]-[7] and the reference given therein).

Our goal in this paper is to investigate the boundedness of the solution of the initial value problem for the second order ordinary differential equation with involution

$$y''(t) = f(t, y(t), y(u(t))), t \in I = (-\infty, \infty), y(t_0) = y_0, y'(t_0) = y'_0. \quad (1)$$

Here and in future $u(t)$ is involution function, that is $u(u(t)) = t$, and t_0 is a fixed point of u . Problem (1) does not seem to yield directly to any techniques that for ordinary differential equations without involution term can be used them in (1). Therefore, we consider the second order linear differential equations with involution. We obtain equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order linear differential equations with involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution is proved. Finally, Theorem on existence and uniqueness of bounded solution of initial value problem for the second order nonlinear ordinary differential equation with involution is established.

2. Linear ordinary differential equation with involution

Let $C^\infty[I]$ be set of all differentiable functions for all degrees.

Theorem 2.1. Let $u(t)$ be a continuous function that it is own inverse on an open interval $I, u \in C^\infty[I]$, and $t_0 \in I$ be a fixed point of u . Let y_0 be any real number and each of $a(t), b(t)$ be a function of class C^∞ on I , such that $b(t)$ does not vanish on the interval I , then the problem

$$y''(t) = a(t)y(t) + b(t)y(-t) + f(t), t \in I, y(0) = \varphi, y'(0) = \psi \tag{2}$$

is equivalent to the following problem for the fourth order ordinary differential equation

$$\begin{cases} y^{(4)}(t) = p(t)y(t) + q(t)y'(t) + r(t)y''(t) + F(t), t \in I, \\ y(0) = \varphi, y'(0) = \psi, \\ y''(0) = a(0)\varphi + b(0)\psi + f(0), \\ y'''(0) = [a'(0) + b'(0)]\varphi + [a(0) - b(0)]\psi + f'(0), \end{cases} \tag{3}$$

where

$$p(t) = a''(t) - \frac{a'(t)}{b(t)} - \frac{2b'(t)a(t)}{b^2(t)} - \frac{a(t)b''(t)}{b(t)} - b(-t)b(t),$$

$$q(t) = 2a'(t) - \frac{a(t)}{b(t)},$$

$$r(t) = -\frac{2(b'(t))^2}{b^2(t)} + \frac{b''(t)}{b(t)} + a(t), s(t) = \frac{1}{b(t)},$$

and

$$F(t) = -\frac{f'(t)}{b(t)} - \frac{2b'(t)f(t)}{b^2(t)} - \frac{b''(t)f(t)}{b(t)} - b(t)f(-t) + f''(t).$$

The proof of Theorem 2.1 is based on approaches of proof of Theorem 1 of paper [?] on the first order linear differential equation with involution.

Now, we consider the initial value problem

$$y''(t) = by(-t) + ay(t) + f(t), t \in I, y(0) = \varphi, y'(0) = \psi \tag{4}$$

for the second order involutory ordinary differential equation. We are interested in studying the stability of problem (4) on I . In general cases of a and b the solution of (4) is not bounded on I . Applying Theorem 2.1, we get the equivalent initial value problem

$$\begin{cases} y^{(4)}(t) + (a^2 - b^2)y(t) - 2ay''(t) = F(t), \\ F(t) = -af(t) + bf(-t) + f''(t), t \in I, \\ y(0) = \varphi, y'(0) = \psi, \\ y''(0) = (b + a)\varphi + f(0), y'''(0) = (-b + a)\psi + f'(0) \end{cases} \tag{5}$$

for the fourth order ordinary differential equation. We will obtain the solution of problem (5). Substituting $m = a^2 - b^2$ inequation $y^{(4)}(t) + (a^2 - b^2)y(t) - 2ay''(t) = F(t)$, we get

$$\begin{cases} y^{(4)}(t) + my(t) - 2ay''(t) = F(t), \\ F(t) = -af(t) + bf(-t) + f''(t), t \in I, \\ y(0) = \varphi, y'(0) = \psi, \\ y''(0) = (b + a)\varphi + f(0), y'''(0) = (-b + a)\psi + f'(0). \end{cases}$$

Using Laplace transform, we get

$$s^4y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + my(s) - 2a(s^2y(s) - sy(0) - y'(0)) = F(s)$$

or

$$(s^4 - 2as^2 + m)y(s) - (s^3 - 2as)y(0) - (s^2 - 2a)y'(0) - sy''(0) - y'''(0) = F(s).$$

Therefore, using initial conditions, we get

$$y(s) = \frac{(s^3 - 2as)\varphi}{s^4 - 2as^2 + m} + \frac{s^2 - 2a}{s^4 - 2as^2 + m}\psi$$

$$+ \frac{s}{s^4 - 2as^2 + m}((b + a)\varphi + f(0))$$

$$+ \frac{(-b + a)\psi + f'(0)}{s^4 - 2as^2 + m} + \frac{F(s)}{s^4 - 2as^2 + m}.$$

Since $s^4 - 2as^2 + m = s^4 - 2as^2 + (a + |b|)(a - |b|) = (s^2 - (a - |b|))(s^2 - (a + |b|))$, we get

$$y(s) = \frac{(s^3 - 2as)\varphi}{(s^2 - (a - |b|))(s^2 - (a + |b|))} + \frac{s^2 - 2a}{s^4 - 2as^2 + m}\psi$$

$$+ \frac{s}{(s^2 - (a - |b|))(s^2 - (a + |b|))}((b + a)\varphi + f(0))$$

$$+ \frac{(-b + a)\psi + f'(0)}{(s^2 - (a - |b|))(s^2 - (a + |b|))} + \frac{F(s)}{(s^2 - (a - |b|))(s^2 - (a + |b|))}.$$

It is easy to see that

$$\frac{1}{(s^2 - (a - |b|))(s^2 - (a + |b|))} \tag{7}$$

$$= -\frac{1}{2|b|} \left[\frac{1}{s^2 - (a - |b|)} - \frac{1}{s^2 - (a + |b|)} \right].$$

Applying the indeterminate coefficients method for fractions, we get

$$\frac{s^3 - 2as}{s^4 - 2as^2 + m} = \frac{As + B}{s^2 - (a - |b|)} + \frac{Cs + D}{s^2 - (a + |b|)}.$$

Then

$$s^3 - 2as = (As + B)(s^2 - (a + |b|)) + (Cs + D)(s^2 - (a - |b|)).$$

Equating the coefficients s^k for $k = 0, 1, 2, 3$, we can write

$$\begin{cases} A + C = 1, B + D = 0, \\ -A(a + |b|) - C(a - |b|) = -2a, \\ -B(a + |b|) - D(a - |b|) = 0. \end{cases}$$

Solving this system of algebraic equations, we get

$$B = D = 0, A = \frac{a + |b|}{2|b|}, C = -\frac{a - |b|}{2|b|}.$$

Therefore,

$$\frac{s^3 - 2as}{s^4 - 2as^2 + m} = \frac{a + |b|}{2|b|} \frac{s}{s^2 - (a - |b|)} - \frac{a - |b|}{2|b|} \frac{s}{s^2 - (a + |b|)}. \tag{8}$$

From formula (7) it follows that

$$\frac{s}{(s^2 - (a - |b|))(s^2 - (a + |b|))} \tag{9}$$

$$= -\frac{1}{2|b|} \left[\frac{s}{s^2 - (a - |b|)} - \frac{s}{s^2 - (a + |b|)} \right].$$

Using formulas (7), (8), and (9), we get

$$y(s) = \left[\frac{a + |b|}{2|b|} \frac{s}{s^2 - (a - |b|)} - \frac{a - |b|}{2b} \frac{s}{s^2 - (a + |b|)} \right] \varphi$$

$$\begin{aligned}
 & + \frac{1}{2|b|} \left[\frac{1}{s^2-(a-|b|)} - \frac{1}{s^2-(a+|b|)} \right] \psi \\
 & + \frac{1}{2|b|} \left[\frac{s}{s^2-(a-|b|)} - \frac{s}{s^2-(a+|b|)} \right] ((b+a)\varphi + f(0)) \\
 & - \frac{1}{2|b|} \left[\frac{1}{s^2-(a-|b|)} - \frac{1}{s^2-(a+|b|)} \right] (f'(0) + (-b+a)\psi) \\
 & - \frac{1}{2|b|} \left[\frac{1}{s^2-(a-|b|)} - \frac{1}{s^2-(a+|b|)} \right] F(s).
 \end{aligned}$$

Taking the inverse Laplace transform, we get

$$\begin{aligned}
 y(t) & = \left[\frac{a+|b|}{2|b|} L^{-1} \left[\frac{s}{s^2-(a-|b|)} \right] - \frac{a-|b|}{2|b|} L^{-1} \left[\frac{s}{s^2-(a+|b|)} \right] \right] \varphi \\
 & + \frac{1}{2|b|} \left[L^{-1} \left[\frac{1}{s^2-(a-|b|)} \right] - L^{-1} \left[\frac{1}{s^2-(a+|b|)} \right] \right] \psi \\
 & + \left[L^{-1} \left[\frac{s}{s^2-(a-|b|)} \right] - L^{-1} \left[\frac{s}{s^2-(a+|b|)} \right] \right] \frac{1}{2|b|} ((b+a)\varphi + f(0)) \\
 & - \frac{1}{2|b|} \left[L^{-1} \left[\frac{1}{s^2-(a-|b|)} \right] - L^{-1} \left[\frac{1}{s^2-(a+|b|)} \right] \right] (f'(0) + (-b+a)\psi) \\
 & - \frac{1}{2|b|} \left[L^{-1} \left[\frac{F(s)}{s^2-(a-|b|)} \right] - L^{-1} \left[\frac{F(s)}{s^2-(a+|b|)} \right] \right].
 \end{aligned}$$

There are three cases: $a \pm |b| > 0$, $a - |b| < 0$ and $a + |b| > 0$, $a \pm |b| < 0$. Let $a \pm |b| > 0$. Then, putting $a \pm |b| = c^2, c > 0$ and applying formulas

$$L^{-1} \left[\frac{s}{s^2-c^2} \right] = \cosh(ct), L^{-1} \left[\frac{1}{s^2-c^2} \right] = \frac{1}{c} \sinh(ct), \tag{10}$$

we get the exact solution of equivalent problems (4) and (5) by formula

$$\begin{aligned}
 y(t) & = \left[\frac{a+|b|}{2|b|} \cosh(\sqrt{a-|b|}t) - \frac{a-|b|}{2|b|} \cosh(\sqrt{a+|b|}t) \right] \varphi \\
 & + \frac{1}{2|b|} \left[\frac{1}{\sqrt{a-|b|}} \sinh(\sqrt{a-|b|}t) - \frac{1}{\sqrt{a+|b|}} \sinh(\sqrt{a+|b|}t) \right] \psi \\
 & + \left[\cosh(\sqrt{a-|b|}t) - \cosh(\sqrt{a+|b|}t) \right] \frac{1}{2b} ((b+a)\varphi + f(0)) \\
 & - \frac{1}{2|b|} \left[\frac{1}{\sqrt{a-|b|}} \sinh(\sqrt{a-|b|}t) - \frac{1}{\sqrt{a+|b|}} \sinh(\sqrt{a+|b|}t) \right] (f'(0) + (-b+a)\psi) \\
 & - \frac{1}{2|b|} \left[\int_0^t \frac{1}{\sqrt{a-|b|}} \sinh(\sqrt{a-|b|}(t-y)) \right. \\
 & \left. - \frac{1}{\sqrt{a+|b|}} \int_0^t \sinh(\sqrt{a+|b|}(t-y)) \right] (-af(y) + bf(-y) + f''(y)) dy.
 \end{aligned} \tag{11}$$

Since $\cosh(ct)$ and $\sinh(ct)$ are not bounded functions on I , in general, we can not obtain stability estimates of the solution for problem (4) for this case.

Let $a \pm |b| < 0$. Then, putting $a \pm |b| = -c^2, c > 0$ and applying formulas

$$L^{-1} \left[\frac{s}{s^2+c^2} \right] = \cos(ct), L^{-1} \left[\frac{1}{s^2+c^2} \right] = \frac{1}{c} \sin(ct), \tag{12}$$

we get the exact solution of equivalent problems (4) and (5) by formula

$$\begin{aligned} y(t) = & \left[\frac{a+|b|}{2|b|} \cos(\sqrt{a-|b|t}) - \frac{a-|b|}{2|b|} \cos(\sqrt{a+|b|t}) \right] \varphi \\ & + \left[\cos(\sqrt{a-|b|t}) - \cos(\sqrt{a+|b|t}) \right] \frac{1}{2|b|} ((b+a)\varphi + f(0)) \\ & + \frac{1}{2|b|} \left[\frac{1}{\sqrt{a-|b|}} \sin(\sqrt{a-|b|t}) - \frac{1}{\sqrt{a+|b|}} \sin(\sqrt{a+|b|t}) \right] \psi \\ & - \frac{1}{2|b|} \left[\frac{1}{\sqrt{a-|b|}} \sin(\sqrt{a-|b|t}) - \frac{1}{\sqrt{a+|b|}} \sin(\sqrt{a+|b|t}) \right] (f'(0) + (-b+a)\psi) \\ & - \frac{1}{2|b|} \int_0^t \left[\frac{1}{\sqrt{a-|b|}} \sin(\sqrt{a-|b|(t-y)}) \right. \\ & \left. - \frac{1}{\sqrt{a+|b|}} \sin(\sqrt{a+|b|(t-y)}) \right] (-af(y) + bf(-y) + f''(y)) dy. \end{aligned} \tag{13}$$

Since $\cos(ct)$ and $\sin(ct)$ are bounded functions on I , we can obtain stability estimates in this case for the solution of problem (4). Note that $a \pm |b| < 0$ is not true for $a \geq 0$. Therefore, in this case we must put $a < 0$. It is clear that from $a \pm |b| < 0$ and $a < 0$ it follows $|b| < |a|$ for $a < 0$.

Let $a - |b| > 0$ and $a + |b| < 0$. Applying formulas (10) and (12), we get

$$\begin{aligned} y(t) = & \left[\frac{a+b}{2b} \cosh(\sqrt{a-bt}) - \frac{a-b}{2b} \cos(\sqrt{a+bt}) \right] \varphi \\ & + [\cosh(\sqrt{a-bt}) - \cos(\sqrt{a+bt})] \frac{1}{2b} ((b+a)\varphi + f(0)) \\ & + \frac{1}{2b} \left[\frac{1}{\sqrt{a-b}} \sinh(\sqrt{a-bt}) - \frac{1}{\sqrt{a+b}} \sin(\sqrt{a+bt}) \right] \psi \\ & \left[-\frac{1}{2b} \frac{1}{\sqrt{a-b}} \sinh(\sqrt{a-bt}) \right. \\ & \left. - \frac{1}{\sqrt{a+b}} \sin(\sqrt{a+bt}) \right] (f'(0) + (-b+a)\psi) \\ & - \frac{1}{2b} \int_0^t \frac{1}{\sqrt{a-b}} \sinh(\sqrt{a-b(t-y)}) F(y) dy \\ & - \frac{1}{\sqrt{a+b}} \int_0^t \sin(\sqrt{a+b(t-y)}) F(y) dy \end{aligned}$$

for $a - |b| < 0$ and $a + |b| > 0$ and

$$\begin{aligned} y(t) = & \left[\frac{a+b}{2b} \cos(\sqrt{a-bt}) - \frac{a-b}{2b} \cosh(\sqrt{a+bt}) \right] \varphi \\ & + [\cos(\sqrt{a-bt}) - \cosh(\sqrt{a+bt})] \frac{1}{2b} ((b+a)\varphi + f(0)) \\ & + \frac{1}{2b} \left[\frac{1}{\sqrt{a-b}} \sin(\sqrt{a-bt}) - \frac{1}{\sqrt{a+b}} \sinh(\sqrt{a+bt}) \right] \psi \\ & \left[-\frac{1}{2b} \frac{1}{\sqrt{a-b}} \sinh(\sqrt{a-bt}) - \frac{1}{\sqrt{a+b}} \sin(\sqrt{a+bt}) \right] (f'(0) + (-b+a)\psi) \end{aligned}$$

$$-\frac{1}{2b} \int_0^t \frac{1}{\sqrt{a-b}} \sinh(\sqrt{a-b}(t-y)) F(y) dy - \frac{1}{\sqrt{a+b}} \int_0^t \sin(\sqrt{a+b}(t-y)) F(y) dy$$

for $a + |b| < 0$ and $a - |b| > 0$. Since $\cosh(ct)$ and $\sinh(ct)$ are not bounded functions on I , in general, we can not obtain stability estimates of the solution for problem (4) for these cases.

Theorem 2.2. Assume that $|b| < |a|$, $a \in (-\infty, 0)$. Then problem (4) is stable and the following stability estimate holds

$$\begin{aligned} \sup_{t \in I} |y(t)| &\leq \frac{1}{2|b|} [|a + |b|| + |a - |b||] |\varphi| + \frac{1}{|b|} [|a + b| |\varphi| + 2|f(0)|] \\ &+ \frac{1}{2|b|} \left[\left| \frac{1}{\sqrt{a-|b|}} \right| + \left| \frac{1}{\sqrt{a+|b|}} \right| \right] [2|f'(0)| + |a - b| |\psi|] \\ &+ \frac{1}{2|b|} \left[\left| \frac{1}{\sqrt{a-|b|}} \right| + \left| \frac{1}{\sqrt{a+|b|}} \right| \right] |\psi| \\ &+ \frac{|a-|b||}{2|b|} \left[\left| \frac{1}{\sqrt{a-|b|}} \right| + \left| \frac{1}{\sqrt{a+|b|}} \right| \right] \int_{-\infty}^{\infty} |f(y)| dy \\ &+ \frac{1}{2|b|} \left[\left| \sqrt{a - |b|} \right| + \left| \sqrt{a + |b|} \right| \right] \int_{-\infty}^{\infty} |f(y)| dy. \end{aligned}$$

The proof is based on formula (13) and the triangle inequality.

3. Nonlinear ordinary differential equation with involution

We consider the initial value problem

$$y''(t) = by(-t) + ay(t) + f(t, y(t), y'(t)), t \in I, y(0) = \varphi, y'(0) = \psi \tag{14}$$

for the second order nonlinear involutory ordinary differential equation. We are interested in studying the existence and uniqueness of bounded solution of problem (14) on I . In general cases of a and b the solution of (14) is not bounded on I . We will apply a fixed point theorem.

Let $C^{(1)}(I)$ be the metric space of all continuously differentiable functions defined on the interval I with the metric d defined by

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)| + \sup_{t \in I} \left| \frac{dx(t)}{dt} - \frac{dy(t)}{dt} \right|.$$

Note that $C^{(1)}(I)$ is the complete space. This is first condition of a fixed point theorem in metric space (see, [9]).

Theorem 3.1. Assume that $|b| < |a|$, $a \in (-\infty, 0)$ and f is continuous and bounded function on the region

$$P = \{(t, x, y) : -\infty < t < \infty, |x - \varphi| < M, |y - \psi| < M\}$$

and $f(0, \varphi, \psi) = 0$, $f_t(0, \varphi, \psi) = 0$. Suppose that f satisfies a Lipschitz condition on P with respect to its second and third arguments, that is, there is a constant l such that for $(t, x, u), (t, y, v) \in P$

$$|f(t, x, u) - f(t, y, v)| \leq l(|x - y| + |u - v|). \tag{15}$$

Then, initial value problem (14) has a unique solution $y \in C^{(1)}(I)$.

Proof. The procedure of proving theorem on the existence and uniqueness of a bounded solution of problem (14) is based on reducing this problem to an integral equation

$$y(t) = Ty(t), \tag{16}$$

where

$$Ty(t) = \left[\frac{a+|b|}{2|b|} \cos(\sqrt{a-|b|}t) - \frac{a-|b|}{2|b|} \cos(\sqrt{a+|b|}t) \right] \varphi$$

$$\begin{aligned}
 & + \left[\cos(\sqrt{a - |b|t}) - \cos(\sqrt{a + |b|t}) \right] \frac{1}{2|b|} (b + a)\varphi \\
 & + \frac{1}{2|b|} \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|t}) - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|t}) \right] \psi \\
 & \quad - \frac{1}{2|b|} \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|t}) \right. \\
 & \quad \left. - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|t}) \right] (-b + a)\psi \\
 & \quad - \frac{1}{2|b|} \int_0^t \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|(t - s)}) \right. \\
 & \quad \left. - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|(t - s)}) \right] \\
 & \quad \times [-af(s, y(s)), y'(s) + bf(-s, y(s)), y'(s)] ds \\
 & \quad + \frac{1}{2|b|} \int_0^t \left[\sqrt{a - |b|} \sin(\sqrt{a - |b|(t - s)}) \right. \\
 & \quad \left. - \sqrt{a + |b|} \sin(\sqrt{a + |b|(t - s)}) \right] f(s, y(s)), y'(s) ds.
 \end{aligned}$$

The proof of equation (16) is based on the formula (13). Note that integral form is a Volterra type integro-differential equation of the second kind. Therefore, the recursive formula for the solution of problem (14) is

$$\begin{aligned}
 y_0(t) &= \left[\frac{a+|b|}{2|b|} \cos(\sqrt{a - |b|t}) - \frac{a-|b|}{2|b|} \cos(\sqrt{a + |b|t}) \right] \varphi \\
 & + \left[\cos(\sqrt{a - |b|t}) - \cos(\sqrt{a + |b|t}) \right] \frac{1}{2|b|} (b + a)\varphi \\
 & + \frac{1}{2|b|} \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|t}) - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|t}) \right] \psi \\
 & \quad - \frac{1}{2|b|} \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|t}) \right. \\
 & \quad \left. - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|t}) \right] (-b + a)\psi, \\
 y_j(t) &= y_0(t) - \frac{1}{2|b|} \int_0^t \left[\frac{1}{\sqrt{a - |b|}} \sin(\sqrt{a - |b|(t - s)}) \right. \\
 & \quad \left. - \frac{1}{\sqrt{a + |b|}} \sin(\sqrt{a + |b|(t - s)}) \right] \\
 & \quad \times [-af(s, y_{j-1}(s)), y'_{j-1}(s) + bf(-s, y_{j-1}(s)), y'_{j-1}(s)] ds \\
 & \quad + \frac{1}{2|b|} \int_0^t \left[\sqrt{a - |b|} \sin(\sqrt{a - |b|(t - s)}) \right. \\
 & \quad \left. - \sqrt{a + |b|} \sin(\sqrt{a + |b|(t - s)}) \right] f(s, y_{j-1}(s)), y'_{j-1}(s) ds, \quad j \geq 1.
 \end{aligned} \tag{17}$$

According to the method of recursive approximation (17), we get

$$y(t) = y_0(t) + \sum_{j=0}^{\infty} [y_{j+1}(t) - y_j(t)]. \tag{18}$$

We have that

$$\begin{aligned}
 y_{j+1}(t) - y_j(t) = & -\frac{1}{2|b|} \int_0^t \left[\frac{1}{\sqrt{a-|b|}} \sin(\sqrt{a-|b|}(t-s)) - \frac{1}{\sqrt{a+|b|}} \sin(\sqrt{a+|b|}(t-s)) \right] \quad (19) \\
 & \times \left[-a[f(s, y_j(s)), y'_j(s)] - f(s, y_{j-1}(s)), y'_{j-1}(s)] + b[f(-s, y_j(s)), y'_j(s)] - f(-s, y_{j-1}(s)), y'_{j-1}(s)] \right] ds \\
 & + \frac{1}{2|b|} \int_0^t \left[\sqrt{a-|b|} \sin(\sqrt{a-|b|}(t-s)) \right. \\
 & \left. - \sqrt{a+|b|} \sin(\sqrt{a+|b|}(t-s)) \right] [f(s, y_j(s)), y'_j(s)] - f(s, y_{j-1}(s)), y'_{j-1}(s)] ds.
 \end{aligned}$$

Therefore, applying the triangle inequality, formula (19) and Lipschitz condition (15), we get

$$|y_{j+1}(t) - y_j(t)|, |y'_{j+1}(t) - y'_j(t)| \leq M(a, b)l \int_{-|t|}^{|t|} [|y_j(s) - y_{j-1}(s)| + |y'_j(s) - y'_{j-1}(s)|] ds \quad (20)$$

for any $t \in I$ and $j \geq 1$. Moreover, applying the triangle inequality, we get

$$|y_0(t)|, |y'_0(t)| \leq M_1(a, b, \varphi, \psi), \quad (21)$$

$$|y_1(t) - y_0(t)|, |y'_1(t) - y'_0(t)| \leq M_2(a, b)|t|, \quad (22)$$

for any $t \in I$. Applying estimates (20) and (22), we can prove that

$$|y_{j+1}(t) - y_j(t)|, |y'_{j+1}(t) - y'_j(t)| \leq [4M(a, b)lM_2(a, b)]^j \frac{|t|^{j+1}}{(j+1)!} \quad (23)$$

for any $t \in I$ and $j \geq 1$. Therefore, applying the triangle inequality, formula (18) and estimates (20) and (23), we get

$$|y(t) - y_n(t)|, |y'(t) - y'_n(t)| \leq \sum_{j=n+1}^{\infty} [4M(a, b)lM_2(a, b)]^j \frac{|t|^{j+1}}{(j+1)!} \rightarrow 0, n \rightarrow \infty,$$

$$|y(t)|, |y'(t)| \leq M_1(a, b, \varphi, \psi) + M_2(a, b)|t| + \sum_{j=1}^{\infty} [4M(a, b)lM_2(a, b)]^j \frac{|t|^{j+1}}{(j+1)!}$$

for any $t \in I$. Theorem 3.1 is proved.

4. Conclusion

In the present paper, the initial value problem for the second order differential equation with involution is investigated. We obtained equivalent initial value problem for the fourth order ordinary differential equations to the initial value problem for second order differential equations with involution. Theorem on stability estimates for the solution of the initial value problem for the second order ordinary linear differential equation with involution is proved. Theorem on existence and uniqueness of bounded solution of initial value problem for the second order ordinary nonlinear differential equation with involution is established. Moreover, applying the result of the monograph [8], the two-step stable difference schemes for the numerical solution of the initial value linear and nonlinear problems (4) and (14) for the second order linear and nonlinear differential equations with involution can be presented and studied.

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